Spherical collapse with heat flow and without horizon

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We present a class of solutions for a heat conducting fluid sphere, which radiates energy during collapse without the appearance of horizon at the boundary at any stage of the collapse. A simple model shows that there is no accumulation of energy due to collapse since it radiates out at the same rate as it is being generated.

One of the interesting problems related to gravitational collapse of stars is to study the solutions of various spherically symmetric fluid distributions with radial heat flux in the interior satisfying all the energy conditions and having reasonable physical behaviour. In this case the exterior spacetime is described by the radiating Vaidya metric [1]. The junction conditions at the boundary of the isotropic fluid sphere with dissipation in the form of the radial heat flux matching with the radiating Vaidya metric were previously studied by Santos [2]. In particular special attention was given to the model which from an initial static configuration starts collapsing when dissipation takes place in the form of heat flux [3]. The initial static configuration evolves until the horizon is formed.

In this letter we present an interesting solution of a collapsing fluid sphere with radial heat flux. The interesting feature of it is that collapse never encounters horizon and it continues until a (naked) singularity is reached. The matter content of this model has reasonable properties and the non-occurrence of the horizon is due to the fact that the rate of mass loss is balanced by the fall of boundary radius. In fact, one gets a class of such solutions although the explicit ex

ample is given for a very simple heat conducting fluid sphere, which satisfies all the energy conditions and has very reasonable physical properties.

The metric in the interior of a shearfree spherically symmetric fluid distribution is given by

$$ds^2 = -A^2(r,t)dt^2 + B^2(r,t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \tag{1}$$

The stress energy tensor of a nonviscous heat conducting fluid reads

$$T^{\mu\nu} = (\rho + p)\, v^\mu v^\nu + p g^{\mu\nu} + q^\mu v^\nu + q^\nu v^\mu. \tag{2}$$

The heat flow vector q^μ is orthogonal to the velocity vector so that $q^\mu v_\mu = 0$. Assuming comoving coordinates $v^\mu = (-g_{00})^{1/2}$ and $q^\mu = (0, q^1, 0, 0)$ nontrivial Einstein equations are [4]

$$\rho = -\frac{4B'}{rB^3} + \frac{3\dot{B}^2}{A^2B^2} + \frac{B'^2}{B^4} - \frac{2B''}{B^3} \tag{3}$$

$$p = \frac{2A'}{rAB^2} + \frac{2B'}{rB^3} + \frac{2\dot{A}\dot{B}}{A^3B} - \frac{\dot{B}^2}{A^2B^2} - \frac{2\ddot{B}}{BA^2} + \frac{2A'B'}{AB^3} + \frac{{B'}^2}{B^4} \tag{4}$$

$$p = \frac{A'}{rAB^2} + \frac{B'}{rB^3} + \frac{2\dot{A}\dot{B}}{A^3B} - \frac{\dot{B}^2}{A^2B^2} - \frac{2\ddot{B}}{BA^2} + \frac{A''}{AB} - \frac{{B'}^2}{B^4} + \frac{B''}{B^3} \tag{5}$$

$$q^1 = -\frac{2A'\dot{B}}{A^2B^3} - \frac{2B'\dot{B}}{AB^4} + \frac{2\dot{B}'}{AB^3} \tag{6}$$

The exterior Vaidya metric already mentioned is given explicitly in the form

$$ds^2 = -\left(1 - \frac{2M(v)}{\overline{r}}\right) dv^2 - 2d\overline{r}dv + \overline{r}^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

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where v is the retarded time and M(v) is the exterior Vaidya mass. The junction conditions then yield the following system of equations valid on the boundary $r = r_\Sigma$ [2],

$$(rB)_\Sigma = \overline{r}_\Sigma \tag{8}$$
$$p_\Sigma = (q^1 B)_\Sigma \tag{9}$$

and

$$m_\Sigma = \left[\frac{r^3 B\dot{B}^2}{2A^2} - r^2 B' - \frac{r^3 B'^2}{2B}\right]_\Sigma \tag{10}$$

where m_Σ is the mass function calculated in the interior at $r = r_\Sigma$ [5, 6]. The junction conditions yield $m_\Sigma = M(v)$.

At this stage we consider a particular form of the metric coefficients given in (1) and choose them separable in r and t co-ordinates as

$$A = a(r) \tag{11}$$
$$B = b(r)R(t) \tag{12}$$

so that ρ , p and q^1 can now be expressed in the form

$$\rho = \frac{1}{R^2}\left[\frac{3}{a^2}\dot{R}^2 - \frac{1}{b^2}\left(\frac{2b''}{b} - \frac{b'^2}{b^2} + \frac{4b'}{rb}\right)\right] \tag{13}$$
$$p = \frac{1}{R^2}\left[-\frac{1}{a^2}(2R\ddot{R} + \dot{R}^2) + \frac{1}{b^2}\left(\frac{b'^2}{b^2} + \frac{2a'b'}{ab} + \frac{2}{r}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right)\right] \tag{14}$$
$$q^1 = -\frac{2a'\dot{R}}{R^3 a^2 b^2} \tag{15}$$

where 'dot' and 'dash' indicate derivatives with respect to time and radial coordinate. The isotropy of pressure would give the equation,

$$\frac{a''}{a} + \frac{b''}{b} - 2\frac{b'^2}{b^2} - 2\frac{a'b'}{ab} - \frac{a'}{ra} - \frac{b'}{rb} = 0. \tag{16}$$

The boundary condition (9) now yields at $r = r_\Sigma = r_o$ in view of (14) and (15)

$$2R\ddot{R} + \dot{R}^2 + m\dot{R} = n \tag{17}$$

where m and n are constants. Such an equation was first given in [7] in the context of a very particular case of Maiti's solution [8]. The general solution of (16) in closed form is not available and a very simple solution is

$$R(t) = -Ct. \tag{18}$$

An interesting feature of the above solution is that at the boundary, $m_\Sigma/\overline{r}\Sigma$ turns out to be independent of time. It can be demonstrated very easily from the expression (10). We then get using the solution (18)

$$\frac{2m_\Sigma}{\overline{r}_\Sigma} = \frac{2m_\Sigma}{(rB)_\Sigma} = 2\left[\frac{C^2 r_0^2 b_0^2}{2a_0^2} - \frac{r_0 b_0'}{b_0} - \frac{r_0^2 b_0'^2}{2b_0^2}\right] \tag{19}$$

where r_0 is written for the comoving radial coordinate at the boundary and $b(r_0) = b_0, b'(r = r_0) = b_0'$. It is quite possible to adjust the aparameters in the above equation so as to keep $2m_\Sigma/\overline{r}_\Sigma < 1$ in order to avoid the appearance of horizon at the boundary.

A simple collapsing model with heat flow but without horizon

We set $b(r) = 1$ in the pressure isotropy equation (16) and consider the special solution [9],

$$A = a(r) = (1 + \xi_0 r^2). \tag{20}$$

and then in view of equations (17-20) we get

$$m = -4\xi_0 r_0,\ n = 4\xi_0(1+\xi_0 r_0^2),\ C = \frac{1}{2}\Big(-|m| + (m^2+4n)^{1/2}\Big). \tag{21}$$

In the solution $R(t) = -Ct$, the constant C is chosen to be positive so that collapsing phase corresponds to $-\infty < t < 0$. We further have

$$C^2 < 4\xi_0(1+\xi_0 r_0^2). \tag{22}$$

The explicit expressions for the density, pressure and heat flow vector are now

$$\rho = \frac{3}{t^2(1+\xi_0 r^2)^2} \tag{23}$$

$$p = \frac{1}{t^2(1+\xi_0 r^2)^2}\left[\frac{4\xi_0}{C^2}(1+\xi_0 r^2) - 1\right] \tag{24}$$

$$q^1 = -\frac{4\xi_0 r}{(1+\xi_0 r^2)^2}\frac{1}{C^2 t^3} \tag{25}$$

All the above quantities diverge at $t \to 0$. The above expressions show that $\rho > 0, p > 0$ and $\rho' < 0$, and $p' < 0$ would require $C^2 < 2\xi_0(1+\xi_0 r^2)$. It would be satisfied throughout the interior if $C^2 < 2\xi_0$. Further $(\rho - p) > 0$ everywhere within the interior implies $C^2 > \xi_0(1+\xi_0 r_0^2)$. All the above physically reasonable conditions will be satisfied provided the following restrictions are imposed on the magnitude of C^2

$$2\xi_0 > C^2 > \xi_0(1+\xi_0 r_0^2). \tag{26}$$

Since there is heat flow in the radial direction the fluid must satisfy another condition in order to be consistent with all the energy conditions and it is $(\rho + p) > 2|q|$, where $|q| \equiv (g_{\mu\nu}q^\mu q^\nu)^{1/2}$. This would require,

$$1 + \frac{2\xi_0}{C^2}(1+\xi_0 r^2) > 4\xi_0 r/C \tag{27}$$

which would always be true as it could be written as

$$\left[1 - \frac{2\xi_0 r}{C}\right]^2 > -\frac{2\xi_0}{C^2}(1-\xi_0 r^2). \tag{28}$$

Now in view of eq. (19), since b = 1 it follows immediately that

$$1 - \frac{2m_\Sigma}{\overline{r}_\Sigma} = \left[1 - \frac{C^2 r_0^2}{(1+\xi_0 r_0^2)^2}\right] \tag{29}$$

So once we have $C^2 < 1/r_0^2 + \xi_0^2 r_0^2 + 2\xi_0$, it is clear that the boundary surface can never reach the horizon. Here in fact both the mass function and the area radius of the radiating sphere vary linearly with t and hence the ratio is independent of time. It may be further noted that though the density, pressure and curvatures would diverge as $1/t^2$ when $t \to 0$, yet the mass function which would go as t , would remain finite and go to zero in this limit. Thus naked singularity would be a weak curvature singalarity. For strong curvature singularity the curvature should diverge at least as t^{-3}. This simple example shows that there is no accumulation of energy due to collapse as it is being radiated out at the same rate as it is being generated.

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